Ohio's Learning Standards | Mathematics

Advanced Quantitative Reasoning-DRAFT

Ohio | Department of Education

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Introduction

PROCESS

To better prepare students for college and careers, educators used public comments along with their professional expertise and experience to revise Ohio's Learning Standards. In spring 2016, the public gave feedback on the standards through an online survey. Advisory committee members, representing various Ohio education associations, reviewed all survey feedback and identified needed changes to the standards. Then they sent their directives to working groups of educators who proposed the actual revisions to the standards. The Ohio Department of Education sent their revisions back out for public comment in July 2016. Once again, the Advisory Committee reviewed the public comments and directed the Working Group to make further revisions. Upon finishing their work, the department presented the revisions to the Senate and House education committees as well as the State Board of Education.

Then in 2019, Ohio started the Strengthening Math Pathways Initiative. Two groups were formed: the Math Pathways Advisory Council and the Math Pathways Architects. The advisory council, made of representatives from education stakeholder groups, aligned systems and structures between secondary and postsecondary mathematics. The Math Pathways Architects, made up of high school and collegiate math faculty, aligned the mathematics between the two systems. One of the outcomes was four proposed Algebra 2 equivalent courses: Quantitative Reasoning, Statistics and Probability, Data Science Foundations, and Discrete Math/Computer Science. A workgroup was formed for each of these courses. This document is the result of the Quantitative Reasoning Workgroup with oversight from the Math Pathways Architects.

UNDERSTANDING MATHEMATICS

These standards define what students should understand and be able to do in their study of mathematics. Asking a student to understand something means asking a teacher to assess whether the student has understood it. But what does mathematical understanding look like? One hallmark of mathematical understanding is the ability to justify, in a way appropriate to the student's mathematical maturity, why a particular mathematical statement is true, or where a mathematical rule comes from. There is a world of difference between a student who can summon a mnemonic device to expand a product such as $(a + b)(x + y)$ and a student who can explain where the mnemonic device comes from. The student who can explain the rule understands the mathematics at a much deeper level. Then the student may have a better chance to succeed at a less familiar task such as expanding $(a + b + c)(x + y)$. Mathematical understanding and procedural skill are equally important, and both are assessable using mathematical tasks of sufficient richness.

The content standards are grade specific. However, they do not define the intervention methods or materials necessary to support students who are well below or well above grade-level expectations. It is also beyond the scope of the standards to define the full range of supports appropriate for English language learners and for students with special needs. At the same time, all students must have the opportunity to learn and meet the same high standards if they are to access the knowledge and skills necessary in their post-school lives. Educators should read the standards allowing for the widest possible range of students to participate fully from the outset. They should provide appropriate accommodations to ensure maximum participation of students with special education needs. For example, schools should allow students with disabilities in reading to use Braille, screen reader technology or other assistive devices. Those with disabilities in writing should have scribes, computers, or speech-to-text technology. In a similar vein, educators should interpret the speaking and listening standards broadly to include sign language. No set of grade-specific standards can fully reflect the great variety in abilities, needs, learning rates, and achievement levels of students in any given classroom.

However, the standards do provide clear signposts along the way to help all students achieve the goal of college and career readiness.

The standards begin on page 10 with the eight Standards for Mathematical Practice.

A Note on Rigor and Algebra 2 Equivalency

Ohio law states that students must have four units of mathematics and that one of those units should be Algebra 2/Math 3 or its equivalent. Ohio has decided to expand guidance around what it means to be *equivalent* to Algebra 2.

It has been decided that *equivalent* refers to the level of rigor and reasoning, not content. There are many branches of mathematics that are equally rigorous but have different content focuses. All equivalent courses should have the same level of rigor and reasoning that are needed to be successful in an entry-level, credit-bearing postsecondary mathematics course.

Ohio has defined rigor as the following:

> "Students use mathematical language to communicate effectively and to describe their work with clarity and precision. Students demonstrate how, when, and why their procedure works and why it is appropriate. Students can answer the question, 'How do we know?'"

This can be illustrated in the table to the right.

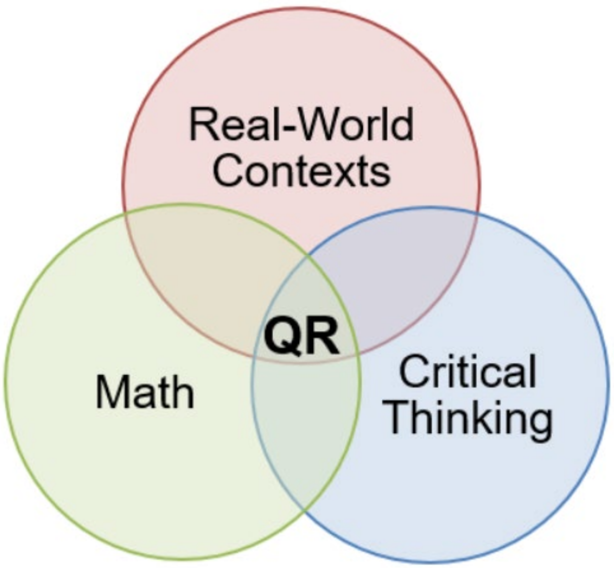
Currently, four courses have been determined to be equivalent to Algebra 2: Advanced Quantitative Reasoning, Statistics and Probability, Data Science Foundations and Discrete Math/Computer Science. This document explains what should be included in an Advanced Quantitative Reasoning course in order to be considered Algebra 2 equivalent. Like a traditional Algebra 2 course, this course should be a year-long course in order for students to earn the credit necessary for graduation.

Rigorous courses are…	vs	Rigorous courses are not…
Defined by complexity, which is a measure of the thinking, action, or knowledge that is needed to complete the task		Characterized by difficulty, which is a measure of effort required to complete a task
Measured in depth of understanding		Measured by the amount of work
Opportunities for precision in reasoning, language, definitions, and notation that are sufficient to appropriate age/course		Based on procedure alone
Determined by students' process		Measured by assigning difficult problems
Opportunities for students to make decisions in problem solving		Defined only by the resources used
Opportunities to make connections		Taught in isolation
Supportive of the transfer of knowledge to new situations		Repetitive
Driven by students developing efficient explanations of solutions and why they work, providing opportunities for thinking and reasoning about contextual problems and situations		Focused on getting an answer
Defined by what the student does with what is given to them		Defined by what is given to the student

What is Advanced Quantitative Reasoning?

DESCRIPTION OF A QUANTIATIVE REASONING COURSE

Quantitative Reasoning is the application of basic mathematics skills, such as algebra, to the analysis and interpretation of quantitative information (numbers and units) in real-world contexts to make decisions relevant to daily life. Critical thinking is its primary objective and outcome. It emphasizes interpretation, representation, calculation, analysis/synthesis, assumptions and communication.



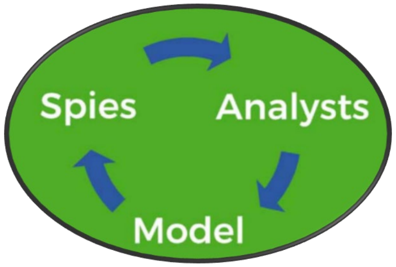
Traditional mathematics courses often look inward to the core of the discipline as they develop formal and/or symbolic skills and abstract reasoning, often using a specially designed language. On the other hand, Quantitative reasoning courses always look outward, aiming to develop practical understanding using plain, straightforward language. Quantitative reasoning is all about promoting practical, robust mathematical habits of the mind.

The six core competencies of quantitative reasoning are the following:

- *Interpretation:* Ability to glean and explain mathematical information presented in various forms (e.g., equations, graphs, diagrams, tables, words).
- *Representation*: Ability to convert information from one mathematical form (e.g., equations, graphs, diagrams, tables, words) into another.
- *Calculation:* Ability to perform arithmetical and mathematical calculations.
- *Analysis/Synthesis:* Ability to make and draw conclusions based on quantitative analysis.
- *Assumptions:* Ability to make and evaluate important assumptions in estimation, modeling, and data analysis.
- *Communication*: Ability to explain thoughts and processes in terms of what evidence is used, how it is organized, presented, and contextualized.

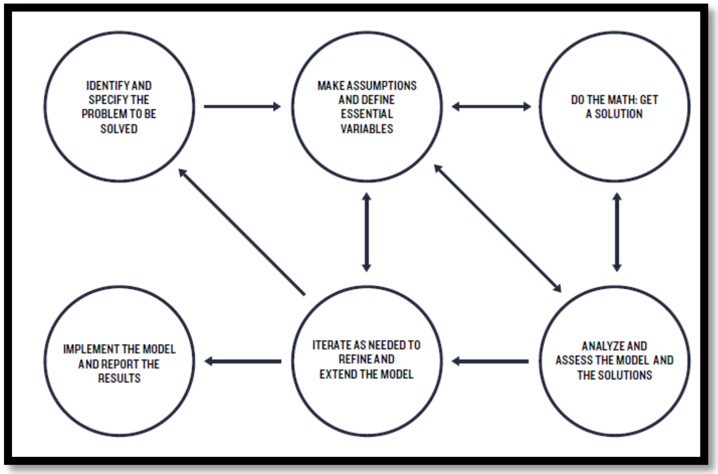
Students should follow a modeling process such as the one describe on pages 20-21 of this document or a similar modeling process such as the Spies, Analysts, Model by Robert Kaplinsky (illustrated below) or the one shown in the GAIMME report (illustrated on the next page).

Robert Kaplinsky's Modeling Process



What is Advanced Quantitative Reasoning?, continued

GAIMME's Modeling Process



Purpose of a Quantitative Reasoning Course

Quantitative literacy is among several important 21st century intellectual skills all students should master, including analytic inquiry, critical and creative thinking, written and oral communication, information literacy, teamwork and problem-solving. In this context, the purposes of a quantitative reasoning course are the following: Strengthen mathematical abilities that students will need in the classroom, in their careers and throughout their lives;

- Engage students in a meaningful intellectual experience that offers them an in-depth understanding of a variety of concepts at a greater depth than a traditional mathematics class;
- Gain the ability to deal with quantitative information as citizens and in the workplace;
- Improve students' quantitative and logical reasoning abilities, allowing them to use a variety of mathematical strategies – breaking difficult questions into component parts, looking at questions from a variety of perspectives and looking for patterns – in diverse settings;
- Improve the ability of students to communicate quantitative ideas orally and in writing; and
- Encourage students to take other courses in the mathematical sciences.

A high school quantitative reasoning course is intended to spark student interest in mathematics by demonstrating connections to the real-world with the purpose of preparing students for a post-high-school learning experience including two-year or four-year college programs, adult career-technical education programs, an apprenticeship and/or the military.

Instruction in a Quantitative Reasoning Course is Different

Appropriate pedagogy is vital to the success of a quantitative reasoning course. The Standards of Mathematical Practices are front and center and are the primary objective of the course. See Appendix A for Ohio Student-Facing Math Practice Rubric. Hiebert and Grouws (2007) researched the development of conceptual understanding; among their findings, the most pertinent to rigorous quantitative reasoning courses are the following:

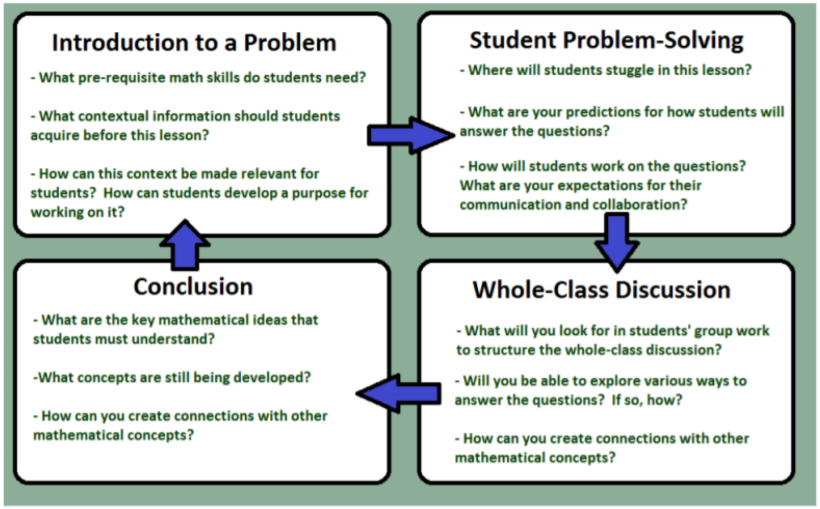
- Students need to be provided with ongoing opportunities to learn.
- Students need to develop deep understanding by forming connections between facts, ideas, and procedures in a social/cultural setting.
- As a product of active discourse, students make connections between mathematical concepts.
- Teachers provide opportunities to learn by allowing students to struggle with grasping important concepts.
- Teachers understand that promoting conceptual understanding also means promoting skill fluency.

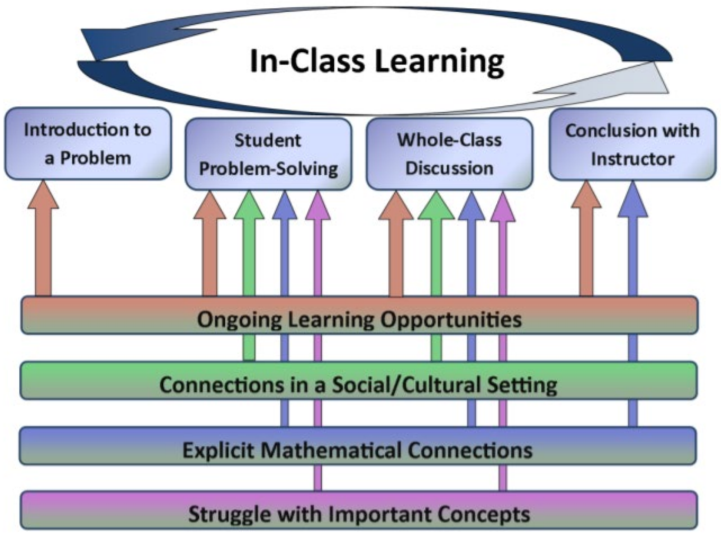
What is Advanced Quantitative Reasoning?, continued

A quantitative reasoning course should look different than a traditional math course as illustrated by the chart below.

Traditional Mathematics Class	Quantitative Reasoning Class
Skill driven with concepts applied	Concept driven with skills applied
Practice and then apply	Apply and then practice
Context removed, solve problem, insert context	Problem cannot be solved without context
Behaviorist approach	Inquiry based
Teacher-centered	Student-centered
Individual-centered	Team-centered
Communication optional	Communication essential
Objects of study or abstract	Objects of study are data
For specific, high-level STEM careers	For all careers, and essential for good citizenship and personal finance decisions

Several authors have presented classroom strategies to promote conceptual learning that are based on stages or cycles defined by Shimizu (1996, 1999). The stages within the problem cycle are designed to create a variety of learning opportunities for students. Each of these steps should be applied multiple times within a class period.





Number talks may also be effectively used to help students reason about mathematics. See Appendix B: Number Talks Emphasizing Mathematical Structure of Fractions for more information on number talks.

What is Advanced Quantitative Reasoning?, continued

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Standards for Mathematical Practice

The Standards for Mathematical Practice describe varieties of expertise that mathematics educators at all levels should seek to develop in their students. These practices rest on important "processes and proficiencies" with longstanding importance in mathematics education. The first of these are the NCTM process standards of problem solving, reasoning and proof, communication, representation, and connections. The second are the strands of mathematical proficiency specified in the National Research Council's report *Adding It Up:* adaptive reasoning, strategic competence, conceptual understanding (comprehension of mathematical concepts, operations and relations), procedural fluency (skill in carrying out procedures flexibly, accurately, efficiently, and appropriately), and productive disposition (habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy).

1. Make sense of problems and persevere in solving them.
Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. They consider analogous problems and try special cases and simpler forms of the original problem in order to gain insight into its solution. They monitor and evaluate their progress and change course if necessary. Older students might, depending on the context of the problem, transform algebraic expressions or change the viewing window on their graphing calculator to get the information they need. Mathematically proficient students can explain correspondences between equations, verbal descriptions, tables, and graphs or draw diagrams of important features and relationships, graph data, and search for regularity or trends. Younger students might rely on using concrete objects or pictures to help conceptualize and solve a problem. Mathematically proficient students check their answers to problems using a different method, and they continually ask themselves, "Does this make sense?" They can understand the approaches of others to solving more complicated problems and identify correspondences between different approaches.

2. Reason abstractly and quantitatively.
Mathematically proficient students make sense of quantities and their relationships in problem situations. They bring two complementary abilities to bear on problems involving quantitative relationships: the ability to *decontextualize*—to abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referents—and the ability to *contextualize*, to pause as needed during the manipulation process in order to probe into the referents for the symbols involved. Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; considering the units involved; attending to the meaning of quantities, not just how to compute them; and knowing and flexibly using different properties of operations and objects.

3. Construct viable arguments and critique the reasoning of others.
Mathematically proficient students understand and use stated assumptions, definitions, and previously established results in constructing arguments. They make conjectures and build a logical progression of statements to explore the truth of their conjectures. They are able to analyze situations by breaking them into cases and can recognize and use counterexamples. They justify their conclusions, communicate them to others, and respond to the arguments of others. They reason inductively about data, making plausible arguments that take into account the context from which the data arose. Mathematically proficient students are also able to compare the effectiveness of two plausible arguments, distinguish correct logic or reasoning from that which is flawed, and—if there is a flaw in an argument—explain what it is. Elementary students can construct arguments using concrete referents such as objects, drawings, diagrams, and actions. Such arguments can make sense and be correct, even though they are not generalized or made formal until later grades. Later, students learn to determine domains to which an argument applies. Students at all grades can listen or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

Standards for Mathematical Practice, continued

4. Model with mathematics.
Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace. In early grades, this might be as simple as writing an addition equation to describe a situation. In middle grades, a student might apply proportional reasoning to plan a school event or analyze a problem in the community.

By high school, a student might use geometry to solve a design problem or use a function to describe how one quantity of interest depends on another. Mathematically proficient students who can apply what they know are comfortable making assumptions and approximations to simplify a complicated situation, realizing that these may need revision later.

They are able to identify important quantities in a practical situation and map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts, and formulas. They can analyze those relationships mathematically to draw conclusions. They routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose.

5. Use appropriate tools strategically.
Mathematically proficient students consider the available tools when solving a mathematical problem. These tools might include pencil and paper, concrete models, a ruler, a protractor, a calculator, a spreadsheet, a computer algebra system, a statistical package, or dynamic geometry software. Proficient students are sufficiently familiar with tools appropriate for their grade or course to make sound decisions about when each of these tools might be helpful, recognizing both the insight to be gained and their limitations. For example, mathematically proficient high school students analyze graphs of functions and solutions generated using a graphing calculator. They detect possible errors by strategically using estimation and other mathematical knowledge. When making mathematical models, they know that technology can enable them to visualize the results of varying assumptions, explore consequences, and compare predictions with data. Mathematically proficient students at various grade levels are able to identify relevant external mathematical resources, such as digital content located on a website, and use them to pose or solve problems. They are able to use technological tools to explore and deepen their understanding of concepts.

6. Attend to precision.
Mathematically proficient students try to communicate precisely to others. They try to use clear definitions in discussion with others and in their own reasoning. They state the meaning of the symbols they choose, including using the equal sign consistently and appropriately. They are careful about specifying units of measure and labeling axes to clarify the correspondence with quantities in a problem. They calculate accurately and efficiently and express numerical answers with a degree of precision appropriate for the problem context. In the elementary grades, students give carefully formulated explanations to each other. By the time they reach high school they have learned to examine claims and make explicit use of definitions.

7. Look for and make use of structure.
Mathematically proficient students look closely to discern a pattern or structure. Young students, for example, might notice that three and seven more is the same amount as seven and three more, or they may sort a collection of shapes according to how many sides the shapes have. Later, students will see 7×8 equals the well-remembered $7 \times 5 + 7 \times 3$, in preparation for learning about the distributive property. In the expression $x^2 + 9x + 14$, older students can see the 14 as 2×7 and the 9 as $2 + 7$. They recognize the significance of an existing line in a geometric figure and can use the strategy of drawing an auxiliary line for solving problems. They also can step back for an overview and shift perspective. They can see complex things, such as some algebraic expressions, as single objects or as being composed of several objects. For example, they can see $5 – 3(x – y)^2$ as 5 minus a positive number times a square and use that to realize that its value cannot be more than 5 for any real numbers x and y.

Standards for Mathematical Practice, continued

8. Look for and express regularity in repeated reasoning.
Mathematically proficient students notice if calculations are repeated and look both for general methods and for shortcuts. Upper elementary students might notice when dividing 25 by 11 that they are repeating the same calculations over and over again, and conclude they have a repeating decimal. By paying attention to the calculation of slope as they repeatedly check whether points are on the line through (1, 2) with slope 3, students might abstract the equation $(y - 2)/(x - 1) = 3$. Noticing the regularity in the way terms cancel when expanding $(x - 1)(x + 1)$, $(x - 1)(x^2 + x + 1)$, and $(x - 1)(x^3 + x^2 + x + 1)$ might lead them to the general formula for the sum of a geometric series. As they work to solve a problem, mathematically proficient students maintain oversight of the process, while attending to the details. They continually evaluate the reasonableness of their intermediate results.

CONNECTING THE STANDARDS FOR MATHEMATICAL PRACTICE TO THE STANDARDS FOR MATHEMATICAL CONTENT

The Standards for Mathematical Practice describe ways in which developing student practitioners of the discipline of mathematics increasingly ought to engage with the subject matter as they grow in mathematical maturity and expertise throughout the elementary, middle and high school years. Designers of curricula, assessments, and professional development should all attend to the need to connect the mathematical practices to mathematical content in mathematics instruction.

The Standards for Mathematical Content are a balanced combination of procedure and understanding. Expectations that begin with the word “understand” are often especially good opportunities to connect the practices to the content. Students who lack understanding of a topic may rely on procedures too heavily. Without a flexible base from which to work, they may be less likely to consider analogous problems, represent problems coherently, justify conclusions, apply the mathematics to practical situations, use technology mindfully to work with the mathematics, explain the mathematics accurately to other students, step back for an overview, or deviate from a known procedure to find a shortcut. In short, a lack of understanding effectively prevents a student from engaging in the mathematical practices. In this respect, those content standards which set an expectation of understanding are potential “points of intersection” between the Standards for Mathematical Content and the Standards for Mathematical Practice. These points of intersection are intended to be weighted toward central and generative concepts in the school mathematics curriculum that most merit the time, resources, innovative energies, and focus necessary to qualitatively improve the curriculum, instruction, assessment, professional development, and student achievement in mathematics.

Mathematical Content Standards for High School

PROCESS

The high school standards specify the mathematics that all students should study in order to be college and career ready. Additional mathematics that students should learn in order to take advanced courses such as calculus, advanced statistics, or discrete mathematics is indicated by (+), as in this example:

> *(+) Represent complex numbers on the complex plane in rectangular and polar form (including real and imaginary numbers).*

All standards without a (+) symbol should be in the common mathematics curriculum for all college and career ready students. Standards with a (+) symbol may also appear in courses intended for all students. However, standards with a (+) symbol will not appear on Ohio's State Tests.

The high school standards are listed in conceptual categories:

- Modeling
- Number and Quantity
- Algebra
- Functions
- Geometry
- Statistics and Probability

Conceptual categories portray a coherent view of high school mathematics; a student's work with functions, for example, crosses a number of traditional course boundaries, potentially up through and including calculus.

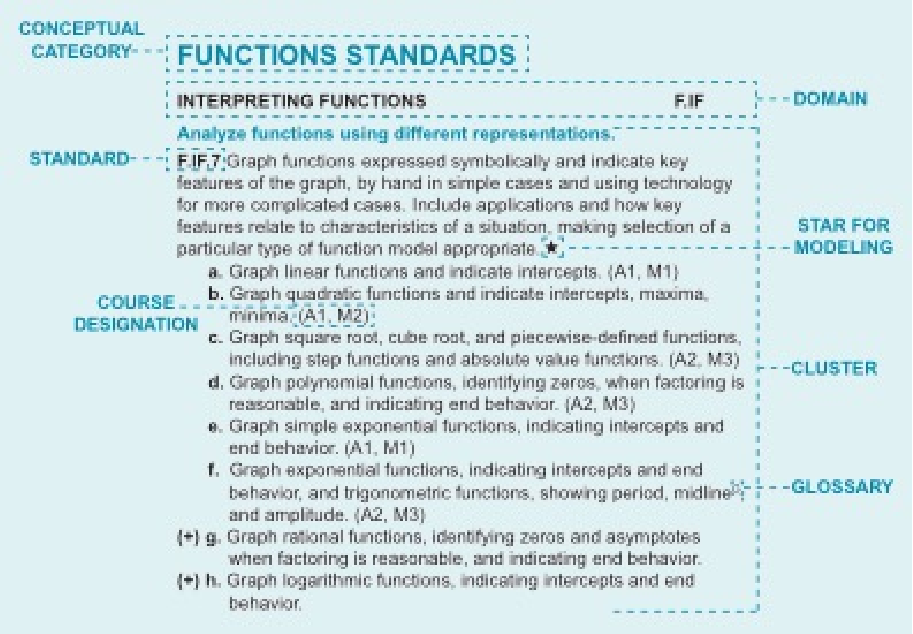
Modeling is best interpreted not as a collection of isolated topics but in relation to other standards. Making mathematical models is a Standard for Mathematical Practice, and specific modeling standards appear throughout the high school standards indicated by a star symbol (★).

Proofs in high school mathematics should not be limited to geometry. Mathematically proficient high school students employ multiple proof methods, including algebraic derivations, proofs using coordinates, and proofs based on geometric transformations, including symmetries. These proofs are supported by the use of diagrams and dynamic software and are written in multiple formats including not just two-column proofs but also proofs in paragraph form, including mathematical symbols. In statistics, rather than using mathematical proofs, arguments are made based on empirical evidence within a properly designed statistical investigation.

How to Read the High School Content Standards

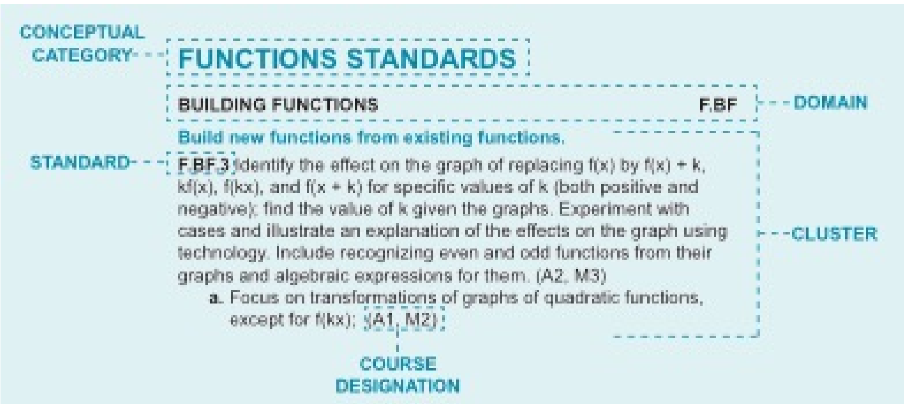
- **Conceptual Categories** are areas of mathematics that cross through various course boundaries.
- **Standards** define what students should understand and be able to do.
- **Clusters** are groups of related standards. Note that standards from different clusters may sometimes be closely related, because mathematics is a connected subject.
- **Domains** are larger groups of related standards. Standards from different domains may sometimes be closely related.
- G shows there is a definition in the glossary for this term.
- (★) indicates that modeling should be incorporated into the standard. (See the Conceptual Category of Modeling pages 60-61)
- (+) indicates that it is a standard for students who are planning on taking advanced courses. Standards with a (+) sign will not appear on Ohio's State Tests.

Some standards have course designations such as (A1, M1) or (A2, M3) listed after an **a., b.,** or **c.** These designations help teachers know where to focus their instruction within the standard. In the example below the beginning section of the standard is the stem. The stem shows what the teacher should be doing for all courses. (Notice in the example below that modeling (★) should also be incorporated.) Looking at the course designations, an Algebra 1 teacher should be focusing his or her instruction on **a.** which focuses on linear functions; **b.** which focuses on quadratic functions; and **e.** which focuses on simple exponential functions. An Algebra 1 teacher can ignore **c., d.,** and **f,** as the focuses of these types of functions will come in later courses. However, a teacher may choose to touch on these types of functions to extend a topic if he or she wishes.



How to Read the High School Content Standards, continued

Notice that in the standard below, the stem has a course designation. This shows that the full extent of the stem is intended for an Algebra 2 or Math 3 course. However, **a.** shows that Algebra 1 and Math 2 students are responsible for a modified version of the stem that focuses on transformations of quadratics functions and excludes the *f(kx)* transformation. However, again a teacher may choose to touch on different types of functions besides quadratics to extend a topic if he or she wishes.



Critical Areas of Focus

CRITICAL AREA OF FOCUS #1

Communication and Analysis

Within this critical area students develop conclusions based on quantitative information and critical thinking. They recognize, make, and evaluate underlying assumptions in estimation, modeling, and data analysis. Students then organize and present thoughts and processes using mathematical and statistical evidence. They communicate clear and complete information in such a way that the reader or listener can understand the contextual and quantitative information in a situation. Students demonstrate numerical reasoning orally and in writing coherent statements and paragraphs.

In the context of real-world applications, students make and investigate mathematical conjectures. They are able to defend their conjectures and respectfully question conjectures made by their classmates. This leads to the development of mathematical arguments and informal proofs, which are ways of expressing particular kinds of reasoning and justification. Explanations (oral and written) include mathematical arguments and rationales, not just procedural descriptions or summaries. Listening to others' explanations gives students opportunities to develop their own understandings. Through communication, ideas become objects of reflection, refinement, discussion, and amendment. When students are challenged to communicate the results of their thinking to others orally or in writing, they learn to be clear, convincing, and precise in their use of mathematical language. Additionally, conversations in which mathematical ideas are explored from multiple perspectives help the participants sharpen their thinking and make connections. **This critical area of focus cross cuts all the other critical areas of focus.**

CRITICAL AREA OF FOCUS #2

Modeling with Number and Quantity

Numeracy is the ability to read, understand and work with quantitative information in order to purposefully use and analyze mathematics in a wide range of situations including in one's personal life, as a citizen and in one's career. Numeracy involves much more than arithmetic skills and straightforward procedural competencies. It involves the ability to be strategic in how and when to use mathematics, understand and interpret mathematical representations, construct and destruct numbers, estimate quantities appropriately, and make sound decisions. Students develop and use the concepts of numeracy to investigate and explain quantitative relationships and solve problems in a variety of real-world contexts. Mathematical concepts involving proportional reasoning such as ratios, rates of change, unit conversions and dimensional analysis are explored. In this course numeracy is applied to a variety of topics including aspects of financial literacy such as budgets and loans. (The teaching of numeracy is intended to both deepen and broaden understanding achieved in previous grades keeping the development and use of all but the most basic algebraic procedures to a minimum. Problems requiring rote use of arithmetic or algebraic procedures should be deemphasized, except when these procedures are essential to gaining deep conceptual understanding.) Modeling in this course should be grounded in concepts outlined in the GAIMME framework[G].

Critical Areas of Focus, continued

CRITICAL AREA OF FOCUS #3

Modeling with Algebra and Functions

The description of modeling as “the process of choosing and using mathematics and statistics to analyze empirical situations, to understand them better, and to make decisions” is at the heart of the course. The narrative discussion and diagram (page 20) of the modeling cycle should be considered when knowledge of functions, statistics, and geometry is applied in a modeling context. Students make decisions by analyzing mathematical models, including situations in which the student must recognize and/or make assumptions. Students synthesize and generalize what they have learned about a variety of function families. They identify appropriate types of functions to model a situation, they adjust parameters to improve the model, and they compare models by analyzing appropriateness of fit and making judgments about the domain over which a model is a good fit. Students model functions in a variety of real-world topics including aspects of financial literacy. Modeling in this course should be grounded in concepts outlined in the GAIMME framework[G].

CRITICAL AREA OF FOCUS #4

Modeling with Statistics and Probability

Students build upon their previous statistical experiences and extend their experience to new situations and deepen their understanding of the statistical process. They use graphical representations and knowledge of the context to make judgments about the appropriateness of statistical models. They develop formal means of assessing how a model fits data[G] such as the use of regression techniques to describe approximately linear relationships between quantities. Students see how the visual displays and summary statistics relate to different types of data and to probability distributions. They identify different ways of collecting data—including sample surveys, experiments, and simulations—and the role that randomness and careful design play in the conclusions that can be drawn. To model the relationships between variables, students fit a linear function to data and analyze residuals to assess the appropriateness of fit. Building on prior experiences associated with the notion of a correlation coefficient, students learn how to distinguish a statistical relationship from a cause-and-effect relationship.

Statistics and Probability in Ohio’s Learning Standards is grounded in GAISE model framework[G]. Within the context of real-world applications, students investigate, represent, make decisions, and draw conclusions about data. Students formulate statistical investigative questions anticipating variability. They acknowledge variability when they collect and consider data. Students will use distributions in accounting for variability when they analyze data. Students interpret the results by looking beyond the data.

CRITICAL AREA OF FOCUS #5

Modeling with Geometry

Students take their prior experience in geometry and apply that learning to real-world problems which could include concepts of congruence, similarity, symmetry, Pythagorean Theorem, circles and trigonometry. They utilize geometric and visual models to represent and understand given scenarios. Geometric methods can be applied to solve real-world design and modeling problems. Of all the subjects students learn in geometry, trigonometry may have the greatest application in college and career. Students in high school should see authentic applications of trigonometry to many different contexts. Graphing utilities, spreadsheets, computer algebra systems, and dynamic geometry software are powerful tools that can be used to model purely mathematical phenomena as well as physical phenomena. Geometric modeling can be used in Fermi[G] problems, problems which ask for rough estimates of quantities.

Students’ experience with two-dimensional and three-dimensional objects may be extended. Students build upon their understanding of area and volume; how changes in dimensions result in similar and non-similar shapes; and how scaling changes lengths, areas and volumes.

Advanced Quantitative Reasoning Course Overview

NUMBER AND QUANTITY

QUANTITIES

- Reason quantitatively and use units to solve problems.
- Reason quantitatively about numerical data including covariation to solve real-world problems. *(proposed)*

ALGEBRA

SEEING STRUCTURE IN EXPRESSIONS

- Interpret the structure of expressions.
- Write expressions in equivalent forms to solve problems.

CREATING EQUATIONS

- Create equations that describe numbers or relationships.

REASONING WITH EQUATIONS AND INEQUALITIES

- Understand solving equations as a process of reasoning and explain the reasoning.
- Solve equations and inequalities in one variable.
- Solve systems of equations.
- Represent and solve equations and inequalities graphically.

MATHEMATICAL PRACTICES

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

FUNCTIONS

INTERPRETING FUNCTIONS

- Understand the concept of a function and use function notation.
- Interpret functions that arise in applications in terms of the context.
- Analyze functions using different representations.

BUILDING FUNCTIONS

- Build a function that models a relationship between two quantities.

LINEAR, QUADRATIC, AND EXPONENTIAL MODELS

- Construct and compare linear, quadratic and exponential models, and solve problems.
- Interpret expressions for functions in terms of the situation they model.

TRIGONOMETRIC FUNCTIONS

- Extend the domain of trigonometric functions using the unit circle.
- Model periodic phenomena with trigonometric functions.

Advanced Quantitative Reasoning Course Overview, continued

GEOMETRY

CONGRUENCE

- Experiment with transformations in the plane.

SIMILARITY, RIGHT TRIANGLES, AND TRIGONOMETRY

- Prove and apply theorems both formally and informally involving similarity using a variety of methods.
- Define trigonometric ratios and solve problems involving right triangles.

CIRCLES

- Find arc lengths and areas of sectors of circles.

GEOMETRIC MEASUREMENT AND DIMENSION

- Understand the relationships between lengths, areas and volumes.

MODELING WITH GEOMETRY

- Apply geometric concepts in modeling situations.

STATISTICS AND PROBABILITY

INTERPRETING CATEGORICAL AND QUANTITATIVE DATA

- Summarize, represent, and interpret data[G] on a single count or measurement variable.
- Summarize, represent, and interpret data on two categorical and quantitative variables.
- Interpret linear models.

MAKING INFERENCES AND JUSTIFYING CONCLUSIONS

- Understand and evaluate random processes underlying statistical experiments.
- Make inferences and justify conclusions from sample surveys, experiments and observational studies.

CONDITIONAL PROBABILITY AND THE RULES OF PROBABILITY

- Understand independence and conditional probability, and use them to interpret data
- Use the rules of probability to compute probabilities of compound events in a uniform probability model.

USING PROBABILITY TO MAKE DECISIONS

- Calculate expected values, and use them to solve problems.
- Use probability to evaluate outcomes of decisions.

High School—Modeling

Modeling links classroom mathematics and statistics to everyday life, work, and decision-making. Modeling is the process of choosing and using appropriate mathematics and statistics to analyze empirical situations, to understand them better, and to improve decisions. Quantities and their relationships in physical, economic, public policy, social, and everyday situations can be modeled using mathematical and statistical methods. When making mathematical models, technology is valuable for varying assumptions, exploring consequences, and comparing predictions with data.

A model can be very simple, such as writing total cost as a product of unit price and number bought or using a geometric shape to describe a physical object like a coin. Even such simple models involve making choices. It is up to us whether to model a coin as a three-dimensional cylinder, or whether a two-dimensional disk works well enough for our purposes. Other situations—modeling a delivery route, a production schedule, or a comparison of loan amortizations—need more elaborate models that use other tools from the mathematical sciences. Real-world situations are not organized and labeled for analysis; formulating tractable models, representing such models, and analyzing them is appropriately a creative process. Like every such process, this depends on acquired expertise as well as creativity.

Some examples of such situations might include the following:

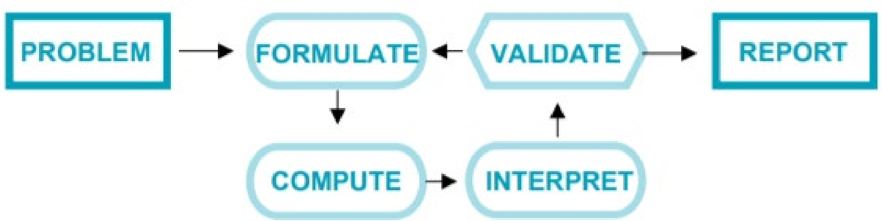
- Estimating how much water and food is needed for emergency relief in a devastated city of 3 million people, and how it might be distributed.
- Planning a table tennis tournament for 7 players at a club with 4 tables, where each player plays against each other player.
- Designing the layout of the stalls in a school fair so as to raise as much money as possible.
- Analyzing stopping distance for a car.
- Modeling savings account balance, bacterial colony growth or investment growth.
- Engaging in critical path analysis, e.g., applied to turnaround of an aircraft at an airport.
- Analyzing risk in situations such as extreme sports, pandemics and terrorism.
- Relating population statistics to individual predictions.

In situations like these, the models devised depend on a number of factors: How precise an answer do we want or need? What aspects of the situation do we most need to understand, control, or optimize? What resources of time and tools do we have? The range of models that we can create and analyze is also constrained by the limitations of our mathematical, statistical, and technical skills, and our ability to recognize significant variables and relationships among them. Diagrams of various kinds, spreadsheets and other technology, and algebra are powerful tools for understanding and solving problems drawn from different types of real-world situations.

High School—Modeling, continued

One of the insights provided by mathematical modeling is that essentially the same mathematical or statistical structure can sometimes model seemingly different situations. Models can also shed light on the mathematical structures themselves, for example, as when a model of bacterial growth makes more vivid the explosive growth of the exponential function.

The basic modeling cycle is summarized in the diagram. It involves (1) identifying variables in the situation and selecting those that represent essential features, (2) formulating a model by creating and selecting geometric, graphical, tabular, algebraic, or statistical representations that describe relationships between the variables, (3) analyzing and performing operations on these relationships to draw conclusions, (4) interpreting the results of the mathematics in terms of the original situation, (5) validating the conclusions by comparing them with the situation, and then either improving the model or, if it is acceptable, (6) reporting on the conclusions and the reasoning behind them. Choices, assumptions, and approximations are present throughout this cycle.



In descriptive modeling, a model simply describes the phenomena or summarizes them in a compact form. Graphs of observations are a familiar descriptive model—for example, graphs of global temperature and atmospheric CO_2 over time.
Analytic modeling seeks to explain data on the basis of deeper theoretical ideas, albeit with parameters that are empirically based; for example, exponential growth of bacterial colonies (until cut-off mechanisms such as pollution or starvation intervene) follows from a constant reproduction rate. Functions are an important tool for analyzing such problems.

Graphing utilities, spreadsheets, computer algebra systems, and dynamic geometry software are powerful tools that can be used to model purely mathematical phenomena, e.g., the behavior of polynomials as well as physical phenomena.

MODELING STANDARDS
Modeling is best interpreted not as a collection of isolated topics but rather in relation to other standards. Making mathematical models is a Standard for Mathematical Practice, and specific modeling standards appear throughout the high school standards indicated by a star symbol (★).

High School—Number and Quantity

NUMBERS AND NUMBER SYSTEMS

During the years from kindergarten to eighth grade, students must repeatedly extend their conception of number. At first, "number" means "counting number": 1, 2, 3... Soon after that, 0 is used to represent "none" and the whole numbers are formed by the counting numbers together with zero. The next extension is fractions. At first, fractions are barely numbers and tied strongly to pictorial representations. Yet by the time students understand division of fractions, they have a strong concept of fractions as numbers and have connected them, via their decimal representations, with the base-ten system used to represent the whole numbers. During middle school, fractions are augmented by negative fractions to form the rational numbers. In Grade 8, students extend this system once more, augmenting the rational numbers with the irrational numbers to form the real numbers. In some high school courses, students will be exposed to yet another extension of number, when the real numbers are augmented by the imaginary numbers to form the complex numbers.

With each extension of number, the meanings of addition, subtraction, multiplication, and division are extended. In each new number system— integers, rational numbers, real numbers, and complex numbers—the four operations stay the same in two important ways: They have the commutative, associative, and distributive properties and their new meanings are consistent with their previous meanings.

Extending the properties of whole number exponents leads to new and productive notation. For example, properties of whole number exponents suggest that $(5^{1/3})^3$ should be $5^{(1/3)\,3} = 5^1 = 5$ and that $5^{1/3}$ should be the cube root of 5.

Calculators, spreadsheets, and computer algebra systems can provide ways for students to become better acquainted with these new number systems and their notation. They can be used to generate data for numerical experiments, to help understand the workings of matrix, vector, and complex number algebra, and to experiment with non-integer exponents.

QUANTITIES

In real-world problems, the answers are usually not numbers but quantities: numbers with units, which involves measurement. In their work in measurement up through Grade 8, students primarily measure commonly used attributes such as length, area, and volume. In high school, students encounter a wider variety of units in modeling, e.g., acceleration, currency conversions, derived quantities such as person hours and heating degree days, social science rates such as per-capita income, and rates in everyday life such as points scored per game or batting averages. They also encounter novel situations in which they themselves must conceive the attributes of interest. For example, to find a good measure of overall highway safety, they might propose measures such as fatalities per year, fatalities per year per driver, or fatalities per vehicle-mile traveled. Such a conceptual process is sometimes called quantification. Quantification is important for science, as when surface area suddenly "stands out" as an important variable in evaporation. Quantification is also important for companies, which must conceptualize relevant attributes and create or choose suitable measures for them.

Number and Quantity Standards

QUANTITIES **N.Q**

Reason quantitatively and use units to solve problems.

N.Q.1 Use units as a way to understand problems and to guide the solution of multi-step problems; choose and interpret units consistently in formulas; choose and interpret the scale and the origin in graphs and data displays. ★

N.Q.2 Define appropriate quantities for the purpose of descriptive modeling.★

N.Q.3 Choose a level of accuracy appropriate to limitations on measurement when reporting quantities.★

Note: Standards N.Q.4-7 are included in this course to be considered for next standards revision.

Reason quantitatively about numerical data including covariation to solve real-world problems.

N.Q.4 *Reason, model, and communicate about numerical data building upon previous knowledge of fractions, decimals, percents, scientific notation and estimation and using that knowledge flexibly in a variety of circumstances and a range of number values.* ★

N.Q.5 *Reason, model, and communicate about proportions including the following:* ★

- ***a.*** *Distinguishing between proportional and non-proportional situations.*
- ***b.*** *Using dimensional analysis.*
- ***c.*** *Analyzing and comparing growth and decay using absolute and relative change.*

N.Q.6 *Use models to solve and communicate about contextual financial questions such as budgets, credit card debt, installment savings, amortization schedules, mortgage and other loan scenarios.*★

N.Q.7 *Identify and explain personal and societal consequences of financial decisions and other scenarios.* ★

High School—Algebra

EXPRESSIONS

An expression is a record of a computation with numbers, symbols that represent numbers, arithmetic operations, exponentiation, and, at more advanced levels, the operation of evaluating a function. Conventions about the use of parentheses and the order of operations assure that each expression is unambiguous. Creating an expression that describes a computation involving a general quantity requires the ability to express the computation in general terms, abstracting from specific instances.

Reading an expression with comprehension involves analysis of its underlying structure. This may suggest a different but equivalent way of writing the expression that exhibits some different aspect of its meaning. For example, $p + 0.05p$ can be interpreted as the addition of a 5% tax to a price p. Rewriting $p + 0.05p$ as $1.05p$ shows that adding a tax is the same as multiplying the price by a constant factor.

Algebraic manipulations are governed by the properties of operations and exponents, and the conventions of algebraic notation. At times, an expression is the result of applying operations to simpler expressions. For example, $p + 0.05p$ is the sum of the simpler expressions p and $0.05p$. Viewing an expression as the result of operation on simpler expressions can sometimes clarify its underlying structure.

A spreadsheet or a computer algebra system (CAS) can be used to experiment with algebraic expressions, perform complicated algebraic manipulations, and understand how algebraic manipulations behave.

EQUATIONS AND INEQUALITIES

An equation is a statement of equality between two expressions, often viewed as a question asking for which values of the variables the expressions on either side are in fact equal. These values are the solutions to the equation. An identity, in contrast, is true for all values of the variables; identities are often developed by rewriting an expression in an equivalent form.

The solutions of an equation in one variable form a set of numbers; the solutions of an equation in two variables form a set of ordered pairs of numbers, which can be plotted in the coordinate plane. Two or more equations and/or inequalities form a system. A solution for such a system must satisfy every equation and inequality in the system.

An equation can often be solved by successively deducing from it one or more simpler equations. For example, one can add the same constant to both sides without changing the solutions but squaring both sides might lead to extraneous solutions. Strategic competence in solving includes looking ahead for productive manipulations and anticipating the nature and number of solutions.

Some equations have no solutions in a given number system but have a solution in a larger system. For example, the solution of $x + 1 = 0$ is an integer, not a whole number; the solution of $2x + 1 = 0$ is a rational number, not an integer; the solutions of $x^2 - 2 = 0$ are real numbers, not rational numbers; and the solutions of $x^2 + 2 = 0$ are complex numbers, not real numbers.

High School—Algebra, continued

The same solution techniques used to solve equations can be used to rearrange formulas. For example, the formula for the area of a trapezoid, $A = \left(\frac{(b_1+b_2)}{2}\right)h$, can be solved for h using the same deductive process.

Inequalities can be solved by reasoning about the properties of inequality. Many, but not all, of the properties of equality continue to hold for inequalities and can be useful in solving them.

CONNECTIONS WITH FUNCTIONS AND MODELING

Expressions can define functions, and equivalent expressions define the same function. Asking when two functions have the same value for the same input leads to an equation; graphing the two functions allows for finding approximate solutions of the equation. Converting a verbal description to an equation, inequality, or system of these is an essential skill in modeling.

Algebra Standards

SEEING STRUCTURE IN EXPRESSIONS A.SSE

Interpret the structure of expressions.

A.SSE.1. Interpret expressions that represent a quantity in terms of its context. ★

- **a.** Interpret parts of an expression, such as terms, factors, and coefficients.
- **b.** Interpret complicated expressions by viewing one or more of their parts as a single entity.

Write expressions in equivalent forms to solve problems.

A.SSE.3 Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.★

(+) **A.SSE.4** Derive the formula for the sum of a finite geometric series (when the common ratio is not 1), and use the formula to solve problems. *For example, calculate mortgage payments.*★

CREATING EQUATIONS A.CED

Create equations that describe numbers or relationships.

A.CED.1 Create equations and inequalities in one variable and use them to solve problems. *Include equations and inequalities arising from linear, quadratic, simple rational, and exponential functions.* ★

- **c.** Extend to include more complicated function situations with the option to solve with technology. (A2, M3)

A.CED.2 Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.★

- **c.** Extend to include more complicated function situations with the option to graph with technology. (A2, M3)

Create equations that describe numbers or relationships, continued.

A.CED.3 Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or non-viable options in a modeling context. *For example, represent inequalities describing nutritional and cost constraints on combinations of different foods.*★ (A1, M1)

- **a.** While functions will often be linear, exponential, or quadratic, the types of problems should draw from more complicated situations. (A2, M3)

A.CED.4 Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations.★

- **d.** While functions will often be linear, exponential, or quadratic, the types of problems should draw from more complicated situations. (A2, M3)

REASONING WITH EQUATIONS AND INEQUALITIES A.REI

Understand solving equations as a process of reasoning and explain the reasoning.

A.REI.1 Explain each step in solving a simple equation as following from the equality of numbers asserted at the previous step, starting from the assumption that the original equation has a solution. Construct a viable argument to justify[G] a solution method.

Solve equations and inequalities in one variable.

A.REI.3 Solve linear equations and inequalities in one variable, including equations with coefficients represented by letters.

Solve systems of equations.

A.REI.6 Solve systems of linear equations algebraically and graphically.

Algebra Standards, continued

REASONING WITH EQUATIONS AND INEQUALITIES, con't A.REI

Represent and solve equations and inequalities graphically.

A.REI.10 Understand that the graph of an equation in two variables is the set of all its solutions plotted in the coordinate plane, often forming a curve (which could be a line).

A.REI.11 Explain why the *x*-coordinates of the points where the graphs of the equation $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$; find the solutions approximately, e.g., using technology to graph the functions, making tables of values, or finding successive approximations.

A.REI.12 Graph the solutions to a linear inequality in two variables as a half-plane (excluding the boundary in the case of a strict inequality), and graph the solution set to a system of linear inequalities in two variables as the intersection of the corresponding half-planes.

High School—Functions

Functions describe situations where one quantity determines another. For example, the return on $10,000 invested at an annualized percentage rate of 4.25% is a function of the length of time the money is invested. Because we continually make theories about dependencies between quantities in nature and society, functions are important tools in the construction of mathematical models.

In school mathematics, functions usually have numerical inputs and outputs and are often defined by an algebraic expression. For example, the time in hours it takes for a car to drive 100 miles is a function of the car's speed in miles per hour, v; the rule $T(v) = {}^{100}/_{v}$ expresses this relationship algebraically and defines a function whose name is T.

The set of inputs to a function is called its domain. We often infer the domain to be all inputs for which the expression defining a function has a value, or for which the function makes sense in a given context.

A function can be described in various ways, such as by a graph, e.g., the trace of a seismograph; by a verbal rule, as in, "I'll give you a state, you give me the capital city;" by an algebraic expression like $f(x) = a + bx$; or by a recursive rule. The graph of a function is often a useful way of visualizing the relationship of the function models and manipulating a mathematical expression for a function can throw light on the function's properties.

Functions presented as expressions can model many important phenomena. Two important families of functions characterized by laws of growth are linear functions, which grow at a constant rate, and exponential functions, which grow at a constant percent rate. Linear functions with a constant term of zero describe proportional relationships.

A graphing utility or a computer algebra system can be used to experiment with properties of these functions and their graphs and to build computational models of functions, including recursively defined functions.

CONNECTIONS TO EXPRESSIONS, EQUATIONS, MODELING, AND COORDINATES.

Determining an output value for a particular input involves evaluating an expression; finding inputs that yield a given output involves solving an equation. Questions about when two functions have the same value for the same input lead to equations, whose solutions can be visualized from the intersection of their graphs. Because functions describe relationships between quantities, they are frequently used in modeling. Sometimes functions are defined by a recursive process, which can be displayed effectively using a spreadsheet or other technology.

Functions Standards

INTERPRETING FUNCTIONS F.IF

Understand the concept of a function, and use function notation.

F.IF.2 Use function notation, evaluate functions for inputs in their domains, and interpret statements that use function notation in terms of a context.

Interpret functions that arise in applications in terms of the context.

F.IF.4 For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. *Key features include the following: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.*★ (A2, M3)

F.IF.5 Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function h(n) gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.*★

- **c.** Emphasize the selection of a type of function for a model based on behavior of data and context. (A2, M3)

F.IF.6 Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph. ★ (A2, M3)

Analyze functions using different representations.

F.IF.7 Graph functions expressed symbolically and indicate key features of the graph, by hand in simple cases and using technology for more complicated cases. Include applications and how key features relate to characteristics of a situation, making selection of a particular type of function model appropriate.★

F.IF.9 Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). *For example, given a graph of one quadratic function and an algebraic expression for another, say which has the larger maximum.* (A2, M3)

BUILDING FUNCTIONS F.BF

Build a function that models a relationship between two quantities.

F.BF.1 Write a function that describes a relationship between two quantities.★

- **a.** Determine an explicit expression, a recursive process, or steps for calculation from context.
- **b.** Combine standard function types using arithmetic operations. *For example, build a function that models the temperature of a cooling body by adding a constant function to a decaying exponential, and relate these functions to the model.* (A2, M3)
- (+) **c.** Compose functions. *For example, if T(y) is the temperature in the atmosphere as a function of height, and h(t) is the height of a weather balloon as a function of time, then T(h(t)) is the temperature at the location of the weather balloon as a function of time.*

F.BF.2 Write arithmetic and geometric sequences both recursively and with an explicit formula, use them to model situations, and translate between the two forms.★

Functions Standards, continued

LINEAR, QUADRATIC, AND EXPONENTIAL MODELS F.LE

Construct and compare linear, quadratic, and exponential models, and solve problems.

F.LE.1 Distinguish between situations that can be modeled with linear functions and with exponential functions.★

- **b.** Recognize situations in which one quantity changes at a constant rate per unit interval relative to another.
- **c.** Recognize situations in which a quantity grows or decays by a constant percent rate per unit interval relative to another.

F.LE.2 Construct linear and exponential functions, including arithmetic and geometric sequences, given a graph, a description of a relationship, or two input-output pairs (include reading these from a table).★

(+)F.LE.3 Observe using graphs and tables that a quantity increasing exponentially eventually exceeds a quantity increasing linearly or quadratically. ★ (A1, M2)

Interpret expressions for functions in terms of the situation they model.

F.LE.5 Interpret the parameters in a linear or exponential function in terms of a context.★

TRIGONOMETRIC FUNCTIONS F.TF

Extend the domain of trigonometric functions using the unit circle.

F.TF.2 Explain how the unit circle in the coordinate plane enables the extension of trigonometric functions to all real numbers, interpreted as radian measures of angles traversed counterclockwise around the unit circle.

Model periodic phenomena with trigonometric functions.

F.TF.5 Choose trigonometric functions to model periodic phenomena with specified amplitude, frequency, and midline[G].★

High School—Geometry

An understanding of the attributes and relationships of geometric objects can be applied in diverse contexts—interpreting a schematic drawing, estimating the amount of wood needed to frame a sloping roof, rendering computer graphics, or designing a sewing pattern for the most efficient use of material.

Although there are many types of geometry, school mathematics is devoted primarily to plane Euclidean geometry, studied both synthetically (without coordinates) and analytically (with coordinates). Euclidean geometry is characterized most importantly by the Parallel Postulate, that through a point not on a given line there is exactly one parallel line. (Spherical geometry, in contrast, has no parallel lines.)

During high school, students begin to formalize their geometry experiences from elementary and middle school, using more precise definitions and developing careful proofs. Later in college some students develop Euclidean and other geometries carefully from a small set of axioms.

The concepts of congruence, similarity, and symmetry can be understood from the perspective of geometric transformation. Fundamental are the rigid motions: translations, rotations, reflections, and combinations of these, all of which are here assumed to preserve distance and angles (and therefore shapes generally). Reflections and rotations each explain a particular type of symmetry, and the symmetries of an object offer insight into its attributes—as when the reflective symmetry of an isosceles triangle assures that its base angles are congruent.

In the approach taken here, two geometric figures are defined to be congruent if there is a sequence of rigid motions that carries one onto the other. This is the principle of superposition. For triangles, congruence means the equality of all corresponding pairs of sides and all corresponding pairs of angles. During the middle grades, through experiences drawing triangles from given conditions, students notice ways to specify enough measures in a triangle to ensure that all triangles drawn with those measures are congruent. Once these triangle congruence criteria (ASA, SAS, and SSS) are established using rigid motions, they can be used to prove theorems about triangles, quadrilaterals, and other geometric figures.

Similarity transformations (rigid motions followed by dilations) define similarity in the same way that rigid motions define congruence, thereby formalizing the similarity ideas of “same shape” and “scale factor” developed in the middle grades. These transformations lead to the criterion for triangle similarity that two pairs of corresponding angles are congruent.

The definitions of sine, cosine, and tangent for acute angles are founded on right triangles and similarity, and, with the Pythagorean Theorem, are fundamental in many real-world and theoretical situations. The Pythagorean Theorem is generalized to non- right triangles by the Law of Cosines. Together, the Laws of Sines and Cosines embody the triangle congruence criteria for the cases where three pieces of information suffice to completely solve a triangle. Furthermore, these laws yield two possible solutions in the ambiguous case, illustrating that Side-Side-Angle is not a congruence criterion.

High School—Geometry, continued

Analytic geometry connects algebra and geometry, resulting in powerful methods of analysis and problem solving. Just as the number line associates numbers with locations in one dimension, a pair of perpendicular axes associates pairs of numbers with locations in two dimensions. This correspondence between numerical coordinates and geometric points allows methods from algebra to be applied to geometry and vice versa. The solution set of an equation becomes a geometric curve, making visualization a tool for doing and understanding algebra. Geometric shapes can be described by equations, making algebraic manipulation into a tool for geometric understanding, modeling, and proof. Geometric transformations of the graphs of equations correspond to algebraic changes in equations.

Dynamic geometry environments provide students with experimental and modeling tools that allow them to investigate geometric phenomena in much the same way as computer algebra systems allow them to experiment with algebraic phenomena.

CONNECTIONS TO EQUATIONS

The correspondence between numerical coordinates and geometric points allows methods from algebra to be applied to geometry and vice versa. The solution set of an equation becomes a geometric curve, making visualization a tool for doing and understanding algebra. Geometric shapes can be described by equations, making algebraic manipulation into a tool for geometric understanding, modeling, and proof.

Geometry Standards

CONGRUENCE **G.CO**

Experiment with transformations in the plane.

G.CO.2 Represent transformations in the plane using, e.g., transparencies and geometry software; describe transformations as functions that take points in the plane as inputs and give other points as outputs. Compare transformations that preserve distance and angle to those that do not, e.g., translation versus horizontal stretch.

G.CO.3 Identify the symmetries of a figure, which are the rotations and reflections that carry it onto itself.

- **a.** Identify figures that have line symmetry; draw and use lines of symmetry to analyze properties of shapes.
- **b.** Identify figures that have rotational symmetry; determine the angle of rotation, and use rotational symmetry to analyze properties of shapes.

G.CO.5 Given a geometric figure and a rotation, reflection, or translation, draw the transformed figure using items such as graph paper, tracing paper, or geometry software. Specify a sequence of transformations that will carry a given figure onto another.

SIMILARITY, RIGHT TRIANGLES, AND TRIGONOMETRY **G.SRT**

Prove and apply theorems both formally and informally involving similarity using a variety of methods.

G.SRT.4 Prove and apply theorems about triangles. *Theorems include but are not restricted to the following: a line parallel to one side of a triangle divides the other two proportionally, and conversely; the Pythagorean Theorem proved using triangle similarity.*

G.SRT.5 Use congruence and similarity criteria for triangles to solve problems and to justify relationships in geometric figures that can be decomposed into triangles.

Define trigonometric ratios, and solve problems involving right triangles.

G.SRT.8 Solve problems involving right triangles.★

CIRCLES **G.C**

Find arc lengths and areas of sectors of circles.

G.C.5 Find arc lengths and areas of sectors of circles.

GEOMETRIC MEASUREMENT AND DIMENSION **G.GMD**

Understand the relationships between lengths, areas, and volumes.

G.GMD.6 When figures are similar, understand and apply the fact that when a figure is scaled by a factor of k, the effect on lengths, areas, and volumes is that they are multiplied by k, k^2, and k^3, respectively.

MODELING WITH GEOMETRY **G.MG**

Apply geometric concepts in modeling situations.

G.MG.1 Use geometric shapes, their measures, and their properties to describe objects, e.g., modeling a tree trunk or a human torso as a cylinder.★

G.MG.2 Apply concepts of density based on area and volume in modeling situations, e.g., persons per square mile, BTUs per cubic foot.★

G.MG.3 Apply geometric methods to solve design problems, e.g., designing an object or structure to satisfy physical constraints or minimize cost; working with typographic grid systems based on ratios.★

High School—Statistics and Probability

Decisions or predictions are often based on data—numbers in context. These decisions or predictions would be easy if the data always sent a clear message, but the message is often obscured by variability. Statistics provides tools for describing variability in data and for making informed decisions that take it into account.

Data are gathered, displayed, summarized, examined, and interpreted to discover patterns and deviations from patterns. Quantitative data can be described in terms of key characteristics: measures of shape, center, and spread. The shape of a data distribution might be described as symmetric, skewed, flat, or bell shaped, and it might be summarized by a statistic measuring center (such as mean or median) and a statistic measuring spread (such as standard deviation or interquartile range). Different distributions can be compared numerically using these statistics or compared visually using plots. Knowledge of center and spread are not enough to describe a distribution. Which statistics to compare, which plots to use, and what the results of a comparison might mean, depend on the question to be investigated and the real-life actions to be taken.

Randomization has two important uses in drawing statistical conclusions. First, collecting data from a random sample of a population makes it possible to draw valid conclusions about the whole population, taking variability into account. Second, randomly assigning individuals to different treatments allows a fair comparison of the effectiveness of those treatments. A statistically significant outcome is one that is unlikely to be due to chance alone, and this can be evaluated only under the condition of randomness. The conditions under which data are collected are important in drawing conclusions from the data; in critically reviewing uses of statistics in public media and other reports, it is important to consider the study design, how the data were gathered, and the analyses employed as well as the data summaries and the conclusions drawn.

Random processes can be described mathematically by using a probability model: a list or description of the possible outcomes (the sample space), each of which is assigned a probability. In situations such as flipping a coin, rolling a number cube, or drawing a card, it might be reasonable to assume various outcomes are equally likely. In a probability model, sample points represent outcomes and combine to make up events; probabilities of events can be computed by applying the Addition and Multiplication Rules. Interpreting these probabilities relies on an understanding of independence and conditional probability, which can be approached through the analysis of two-way tables.

Technology plays an important role in statistics and probability by making it possible to generate plots, regression functions, and correlation coefficients, and to simulate many possible outcomes in a short amount of time.

CONNECTIONS TO FUNCTIONS AND MODELING

Functions may be used to describe data; if the data suggest a linear relationship, the relationship can be modeled with a regression line, and its strength and direction can be expressed through a correlation coefficient.

Statistics and Probability Standards

INTERPRETING CATEGORICAL AND QUANTITATIVE DATA S.ID

Summarize, represent, and interpret data[G] on a single count or measurement variable.

S.ID.1 Represent data with plots on the real number line (dot plots[G], histograms, and box plots[G]) in the context of real-world applications using the GAISE model.★

S.ID.2 In the context of real-world applications by using the GAISE model, use statistics appropriate to the shape of the data distribution to compare center (median and mean) and spread (mean absolute deviation[G], interquartile range[G], and standard deviation) of two or more different data sets. ★

S.ID.3 In the context of real-world applications by using the GAISE model, interpret differences in shape, center, and spread in the context of the data sets, accounting for possible effects of extreme data points (outliers). ★

S.ID.4 Use the mean and standard deviation of a data set to fit it to a normal distribution and to estimate population percentages. Recognize that there are data sets for which such a procedure is not appropriate. Use calculators, spreadsheets, and tables to estimate areas under the normal curve.★

Summarize, represent, and interpret data on two categorical and quantitative variables.

S.ID.6 Represent data on two quantitative variables on a scatter plot[G], and describe how the variables are related.★

- **a.** Fit a function to the data; use functions fitted to data to solve problems in the context of the data. *Use given functions, or choose a function suggested by the context. Emphasize linear, quadratic, and exponential models.* (A2, M3)
- **b.** Informally assess the fit of a function by discussing residuals. (A2, M3)
- **c.** Fit a linear function for a scatterplot that suggests a linear association. (A1, M1)

Interpret linear models.

S.ID.7 Interpret the slope (rate of change) and the intercept (constant term) of a linear model in the context of the data.★

S.ID.8 Compute (using technology) and interpret the correlation coefficient of a linear fit.★

S.ID.9 Distinguish between correlation and causation.★

MAKING INFERENCES AND JUSTIFYING CONCLUSIONS S.IC

Understand and evaluate random processes underlying statistical experiments.

S.IC.1 Understand statistics as a process for making inferences about population parameters based on a random sample from that population.★

S.IC.2 Decide if a specified model is consistent with results from a given data-generating process, e.g., using simulation. *For example, a model says a spinning coin falls heads up with probability 0.5. Would a result of 5 tails in a row cause you to question the model?*★

Make inferences and justify conclusions from sample surveys, experiments, and observational studies.

S.IC.3 Recognize the purposes of and differences among sample surveys, experiments, and observational studies; explain how randomization relates to each.★

S.IC.4 Use data from a sample survey to estimate a population mean or proportion; develop a margin of error through the use of simulation models for random sampling.★

Statistics and Probability Standards, cont'd

MAKING INFERENCES AND JUSTIFYING CONCLUSIONS, cont'd S.IC

Make inferences and justify conclusions from sample surveys, experiments, and observational studies, continued.

S.IC.5 Use data from a randomized experiment to compare two treatments; use simulations to decide if differences between sample statistics are statistically significant.★

S.IC.6 Evaluate reports based on data.★

CONDITIONAL PROBABILITY AND THE RULES OF PROBABILITY[G] S.CP

Understand independence and conditional probability, and use them to interpret data.

S.CP.3 Understand the conditional probability of A given B as $^{P(A \text{ and } B)}/_{P(B)}$, and interpret independence of A and B as saying that the conditional probability of A given B is the same as the probability of A, and the conditional probability of B given A is the same as the probability of B.★

S.CP.4 Construct and interpret two-way frequency tables of data when two categories are associated with each object being classified. Use the two-way table as a sample space to decide if events are independent and to approximate conditional probabilities. *For example, collect data from a random sample of students in your school on their favorite subject among math, science, and English. Estimate the probability that a randomly selected student from your school will favor science given that the student is in tenth grade. Do the same for other subjects and compare the results.*★

Understand independence and conditional probability, and use them to interpret data, continued.

S.CP.5 Recognize and explain the concepts of conditional probability and independence in everyday language and everyday situations. *For example, compare the chance of having lung cancer if you are a smoker with the chance of being a smoker if you have lung cancer.*★

Use the rules of probability to compute probabilities of compound events in a uniform probability model[G].

S.CP.6 Find the conditional probability of A given B as the fraction of B's outcomes that also belong to A, and interpret the answer in terms of the model.★

S.CP.7 Apply the Addition Rule, P(A or B) = P(A) + P(B) − P(A and B), and interpret the answer in terms of the model.★

USING PROBABILITY TO MAKE DECISIONS S.MD

Calculate expected values, and use them to solve problems.

S.MD.1 Define a random variable[G] for a quantity of interest by assigning a numerical value to each event in a sample space; graph the corresponding probability distribution[G] using the same graphical displays as for data distributions.★

S.MD.2 Calculate the expected value[G] of a random variable; interpret it as the mean of the probability distribution.★

Calculate expected values, and use them to solve problems.

S.MD.3 Develop a probability distribution for a random variable defined for a sample space in which theoretical probabilities can be calculated; find the expected value. *For example, find the theoretical probability distribution for the number of correct answers obtained by guessing on all five questions of a multiple-choice test where each question has four choices, and find the expected grade under various grading schemes.*★

Statistics and Probability Standards, cont'd

USING PROBABILITY TO MAKE DECISIONS, continued S.MD

Calculate expected values, and use them to solve problems, continued.

S.MD.4 Develop a probability distribution for a random variable defined for a sample space in which probabilities are assigned empirically; find the expected value. *For example, find a current data distribution on the number of TV sets per household in the United States, and calculate the expected number of sets per household. How many TV sets would you expect to find in 100 randomly selected households?*★

Use probability to evaluate outcomes of decisions.

S.MD.5 Weigh the possible outcomes of a decision by assigning probabilities to payoff values and finding expected values.★

- **a.** Find the expected payoff for a game of chance. *For example, find the expected winnings from a state lottery ticket or a game at a fast-food restaurant.*
- **b.** Evaluate and compare strategies on the basis of expected values. *For example, compare a high-deductible versus a low-deductible automobile insurance policy using various, but reasonable, chances of having a minor or a major accident.*

S.MD.6 Use probabilities to make fair decisions, e.g., drawing by lots, using a random number generator.★

Use probability to evaluate outcomes of decisions, continued.

S.MD.7 Analyze decisions and strategies using probability concepts, e.g., product testing, medical testing, pulling a hockey goalie at the end of a game.★

Glossary

Bivariate data. Pairs of linked numerical observations. Example: a list of heights and weights for each player on a football team.

Box plot. A method of visually displaying a distribution of data values by using the median, quartiles, and extremes of the data set. A box shows the middle 50% of the data.[1] *See also:* first quartile and third quartile.

Congruent. Two plane or solid figures are congruent if one can be obtained from the other by rigid motion (a sequence of rotations, reflections, and translations).

Data. Factual information (such as numbers, measurements, statistics or images) used as a basis for reasoning, decision-making, discussion, or calculation.

Dot plot. *See also:* line plot.

Expected value. For a random variable, the weighted average of its possible values, with weights given by their respective probabilities.

Fermi problem. An estimation problem based on magnitude invented by Italian physicist Enrico Fermi. It is designed to teach dimensional analysis or approximation of quantities that are difficult or impossible to measure.

First quartile. For a data set with median M, the first quartile is the median of the data values less than M. Example: For the data set {1, 3, 6, 7, 10, 12, 14, 15, 22, 120}, the first quartile is 6.2 *See also:* median, third quartile, interquartile range.

GAISE Model *See also:* Pre-K-12 Guidelines for Assessment and Instruction in Statistics Education II (GAISE II): A Framework for Statistics and Data Science Education

Independently combined probability models. Two probability models are said to be combined independently if the probability of each ordered pair in the combined model equals the product of the original probabilities of the two individual outcomes in the ordered pair.

Guidelines for Assessment and Instruction in Mathematics Modeling Education (GAIMME). It is a report written by Consortium for Mathematics and its Applications (COMAP) and the Society of Industrial and Applied Mathematics (SIAM) that provides educators a framework and guideline for incorporating modeling into their classrooms. The report can be found at https://www.comap.com/Free/GAIMME/index.html

Interquartile range. A measure of variation in a set of numerical data, the interquartile range is the distance between the first and third quartiles of the data set. Example: For the data set {1, 3, 6, 7, 10, 12, 14, 15, 22, 120}, the interquartile range is 15 − 6 = 9. *See also:* first quartile, third quartile.

Justify: To provide a convincing argument for the truth of a statement to a particular audience.

Line plot. A method of visually displaying a distribution of data values where each data value is shown as a dot or mark above a number line. Also known as a dot plot.[3]

Mean absolute deviation. A measure of variation in a set of numerical data, computed by adding the distances between each data value and the mean, then dividing by the number of data values. Example: For the data set {2, 3, 6, 7, 10, 12, 14, 15, 22, 120}, the mean absolute deviation is 20.

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[1] Adapted from Wisconsin Department of Public Instruction, http://dpi.wi.gov/ standards/mathglos.html, accessed March 2, 2010.
[2] Many different methods for computing quartiles are in use. The method defined here is sometimes called the Moore and McCabe method. See Langford, E., "Quartiles in Elementary Statistics," Journal of Statistics Education Volume 14, Number 3 (2006).
[3] Adapted from Wisconsin Department of Public Instruction, *op. cit.*
[4] Adapted from Wisconsin Department of Public Instruction, *op. cit.*

Midline. In the graph of a trigonometric function, the horizontal line halfway between its maximum and minimum values

Percent rate of change. A rate of change expressed as a percent. Example: if a population grows from 50 to 55 in a year, it grows by $^{5}/_{50}$ = 10% per year.

Pre-K-12 Guidelines for Assessment and Instruction in Statistics Education II (GAISE II): A Framework for Statistics and Data Science Education. It is an updated report endorsed by the American Statistical Association (ASA) and the National Council of Teachers of Mathematics (NCTM) to enhance the statistics standards. Like the GAISE I, it provides a framework of recommendations for developing students' foundational skills in statistical reasoning in three levels across the school years, described as levels A, B, and C. GAISE I and GAISE II can be found at https://www.amstat.org/asa/education/Guidelines-for-Assessment-and-Instruction-in-Statistics-Education-Reports.aspx.

Probability distribution. The set of possible values of a random variable with a probability assigned to each.

Probability. A number between 0 and 1 used to quantify likelihood for processes that have uncertain outcomes (such as tossing a coin, selecting a person at random from a group of people, tossing a ball at a target, or testing for a medical condition).

Probability model. A probability model is used to assign probabilities to outcomes of a chance process by examining the nature of the process. The set of all outcomes is called the sample space, and their probabilities sum to 1. *See also:* uniform probability model.

Prove: To provide a logical argument that demonstrates the truth of a statement. A proof is typically composed of a series of justifications, which are often single sentences, and may be presented informally or formally.

Random variable. An assignment of a numerical value to each outcome in a sample space.

Rigid motion. A transformation of points in space consisting of a sequence of one or more translations, reflections, and/or rotations. Rigid motions are here assumed to preserve distances and angle measures.

Sample space. In a probability model for a random process, a list of the individual outcomes that are to be considered.

Scatter plot. A graph in the coordinate plane representing a set of bivariate data. For example, the heights and weights of a group of people could be displayed on a scatter plot.[4]

Similarity transformation. A rigid motion followed by a dilation.

Third quartile. For a data set with median *M*, the third quartile is the median of the data values greater than *M*. Example: For the data set {2, 3, 6, 7, 10, 12, 14, 15, 22, 120}, the third quartile is 15. *See also:* median, first quartile, interquartile range.

Uniform probability model. A probability model which assigns equal probability to all outcomes. *See also:* probability model.

Vector. A quantity with magnitude and direction in the plane or in space, defined by an ordered pair or triple of real numbers.

Verify: To check the truth or correctness of a statement in specific cases.

Appendix A: Ohio's Student-Facing Math Practice Rubric

SMP	Novice	Apprentice	Practitioner	Expert
MP.1 Make Sense of Problems and Persevere in Solving Them	Has difficulty working independently or collaboratively; requires continual assistance from teacher or other students.	Works independently and collaboratively with some reliance on the teacher or other students.	Works independently and collaboratively with little reliance on the teacher.	Works independently and leads group in collaboration.
	Gives up easily; rarely takes initiative to find new information or methodologies to solve a problem.	Gives up occasionally; persists only when comfortable; occasionally takes initiative to find some new information or one methodology to solve a problem.	Generally shows some persistence; may give up when extremely challenged; comfortable seeking new information or exploring one or two methodologies to solve a problem.	Continues working when challenged, regularly taking initiative to find new information; fluid in using multiple methodologies and approaches to solve a problem.
	Has difficulty getting started on a problem.	Finds an entry point to a problem.	Finds one or two entry points to a problem.	Discovers multiple entry points to problems.
	Focuses on answers and procedures instead of problem; has difficulty understanding the question of interest being asked; has difficulty analyzing varying aspects of the problem.	Tries to make sense of the problem but may need prompting by classmates or teacher; tries something even if not correct; analyzes some of the aspects of the problem.	Makes sense of the problem; breaks down problems into simpler pieces; articulates the problem into their own words; analyzes information, makes conjectures and plans a solution pathway.	Makes sense and breaks down problems, synthesizing the results when presenting solutions; connects known mathematical ideas and procedures to real-world situations; makes conjectures and plans a solution pathway based on analysis.
	Struggles or forgets to monitor or evaluate progress; Forgets to check their work or evaluate his or her answer to see if it makes sense.	Occasionally monitors and evaluates the progress; may check and revise work by plugging numbers in; may forget to check for reasonableness.	Regularly monitors and evaluates the progress; checks work and determines reasonableness of solution within context and makes necessary modifications.	Monitors and evaluates the progress, changing course as necessary; checks reasonableness; defends and justifies solution by comparing solution paths.
	Does not answer the question of interest.	Partially answers the question of interest.	Thoroughly answers the question of interest.	Thoroughly answers the question and poses new question(s); generalizes.

SMP	Novice	Apprentice	Practitioner	Expert
MP.2 Reason Abstractly and Quantitatively.	Has difficulty understanding that quantities have meaning.	Understands that quantities having meaning but has difficulty seeing all the relationships between quantities.	Attends to the meaning of quantities and the relationships between quantities, rather than simply computing them.	Attends to the meaning of quantities and the relationships between quantities, rather than simply computing them; generalizes their relationships to new situations.
	Ignores units.	Considers the units involved but struggles to understand how they relate to the problem; may use units as labels.	Pays attention to units and uses them appropriately.	Analyzes the use of symbols, variables and units accurately, consistently and appropriately.
	Uses memorized procedures and algorithms incorrectly; cannot represent a situation symbolically.	Uses memorized procedures and algorithms but struggles to explain underlying concepts; can represent and manipulate simple situations symbolically.	Sees connections between procedures or algorithms and contextual meaning but has difficulty articulating them; can represent and manipulate routine situations symbolically.	Articulates connections between procedures or algorithms and contextual meaning; can take a situation and flexibly represent it symbolically and then manipulate the symbols (decontextualize).
	Has difficulty making connections between the symbolic representation and the context.	Sees a connection between the symbolic representation and the context but has difficulty articulating it.	Takes a symbolic situation and relates it to the context (contextualize).	Takes a symbolic situation and relates it to the context and other similar contexts.
	Struggles or forgets to use representations when approaching a problem.	Uses some representations to represent the problem.	Creates a coherent representation of the problem.	Creates, interprets, models, and connects multiple representations coherently.
	Struggles or forgets to finish calculations or interpret solutions.	Finishes calculations but struggles or forgets to interpret solutions.	Interprets solutions appropriately.	Looks beyond solutions to make predictions, generalizations, and decisions.

SMP	Novice	Apprentice	Practitioner	Expert
MP.3 Construct Viable Arguments and Critique the Reasoning of Others	Struggles or forgets to offer any explanation.	Explains reasoning, but does not use mathematical language; attempts to use an object, graph, chart, etc.	Explains and defends reasonings using mathematical language; explains each step and the reasoning behind it using objects, graphs, charts, etc.	Articulates and defends reasoning using concise mathematical language using objects, graphs, charts, etc.
	Forgets to look for errors; arguments are based on opinion.	Identifies some errors; uses some arguments and logic, but arguments may have holes.	Analyzes a problem for errors and misconceptions; uses logical mathematical arguments to support and justify solutions.	Analyzes a problem for misconceptions and errors in logic, computation, and methodologies; uses logical arguments and justifies conclusions; uses definitions and previously established results in constructing arguments.
	Struggles or forgets to defend or give rationale for the choices they made.	Defends their own choices, but concludes which plan is best without solid support, or concludes a single plan is best rather than looking at the conditions under which each plan is best.	Defends their own choices; concludes their choice is best by looking at the conditions of each plan.	Defends their own choices; concludes their choice is best by looking at the conditions of each plan and gives evidence to support their choices.
	Struggles or forgets to seek understanding of classmates' solutions.	Asks for clarification of classmates' solutions.	Questions classmates' solutions and asks "Why?"	Asks classmates probing questions to clarify and improve their arguments.
	Struggles or forgets to list assumptions or logical conjectures.	Lists assumptions and logical conjectures but may not be able to clearly differentiate between them.	Differentiates between assumptions and logical conjectures.	Makes, recognizes and challenges assumptions and logical conjectures by providing evidence; evaluates peer arguments; able to compare the effectiveness of two plausible arguments.

SMP	Novice	Apprentice	Practitioner	Expert
MP.4 Model with Mathematics.	Forgets that attempting to make a simpler problem is a viable strategy.	Attempts to make a simpler problem; struggles to or forgets to identify assumptions.	Uses and identifies assumptions and approximations to make a simpler problem.	Uses and identifies, assumptions and approximations to make a simpler problem; explains how assumptions and/or approximations affect their solution.
	Struggles or forgets to make a model or chooses an inappropriate model.	Uses a model to describe patterns or trends; it may be incomplete or partial model; explains the appropriateness of their model.	Identifies important quantities and maps their relationships using tools such as diagrams, two-way tables, graphs, flowcharts, and/or formulas, etc.; uses their model to describe patterns or trends; evaluates and explains appropriateness of their model.	Creates a model based upon important quantities; evaluates and explains the appropriateness of their model including variables and procedures; reflects on their model and makes any necessary revisions.
	Struggles or forgets to identify variables.	Can correctly identify some, but not all, of the variables.	Identifies and justifies choice of variables in the problem; identifies extraneous or missing information.	Identifies and justifies choice of all variables in the problem; identifies extraneous information; seeks out missing information.
	Struggles or forgets to compute results correctly; has difficulty interpreting their results.	Computes results correctly, but interprets their results, incorrectly, or interprets their results but computes incorrectly.	Computes and interprets results correctly and reports findings using a mixture of representations.	Computes and interprets results and justifies the reasonableness of their results and procedures within the task's context.
	Struggles or forgets to check work.	Checks some calculations.	Checks calculations and checks to see if an answer makes sense within the context of a situation.	Checks to see if an answer makes sense within the context of a situation and changes a model when necessary.

SMP	Novice	Apprentice	Practitioner	Expert
MP.5 Use appropriate tools strategically.	Needs teacher or classmate to instruct them on which tool to choose.	Selects an appropriate tool but is unsure how to use the tool to complete the task. Able to use the tool with some assistance.	Makes sound decisions about the use of specific tools; chooses and uses tools including technology independently to complete tasks.	Makes sound decisions about multiple tools including technology flexibly to complete tasks finding appropriate alternatives when typical tools are not available.
	Struggles or forgets to consider whether or tool is effective or has limitations.	Evaluates some of the effectiveness or limitations of a tool.	Thoroughly evaluates effectiveness or limitations of a tool.	Evaluates effectiveness or limitation or limitations of a tool and leverages that knowledge when analyzing a solution to a problem.
	Has difficulty using tools including technology correctly or efficiently; has difficulty using tools appropriately in exploration of concepts	Uses tools including technology correctly but inefficiently; uses tools appropriately to explore understanding of concepts but struggles to reach conclusions.	Uses tools including technology correctly, efficiently, and strategically; uses technological tools appropriately to explore understanding of concepts and is able to reach conclusions.	Uses tools including technology correctly, efficiently, and strategically; uses technological tools to deepen understanding of concepts, visualize the results of assumptions, explore consequences, and compare predictions with data.
	Struggles or forgets to consider when a solution provided by technology does not make sense.	Realizes when a solution provided by technology does not make sense.	Evaluates the results produced by technology to detect possibly errors by strategically using estimation and other mathematical knowledge.	Evaluates the results produced by technology to detect possibly errors, adjusting the tool, revising solutions or choosing other technological platforms when necessary.

SMP	Novice	Apprentice	Practitioner	Expert
MP.6 Attend to Precision.	Struggles or forgets to address the question completely or directly; gives incomplete or wrong responses.	Makes an incomplete attempt to answer the question; struggles or forgets to formulate further responses when necessary.	Answers the question of interest thoroughly and precisely; provides carefully formulated explanations; revises responses when necessary.	Answers the question of interest thoroughly and precisely; provides carefully formulated explanations; identifies when others are not addressing the question completely.
	Has difficulty communicating; struggles to decipher appropriate depth of communication needed, e.g., goes off on tangents or does not provide relevant information.	Communicates with vague or incorrect terminology.	Communicates clearly with a good grasp of the mathematical terminology. uses mathematical vocabulary carefully.	Communicates clearly and concisely, examining claims and making explicit use of mathematical definitions.
	Calculates inaccurately; forgets to specify the meaning of symbols and units and to provide labels.	Calculates accurately but not precisely, occasionally specifying the meaning of symbols and units and providing labels.	Calculates accurately and efficiently, expressing numerical answers with a degree of precision; states the meaning of symbols, specifying units of measure and providing accurate labels.	Calculates accurately and efficiently and uses proper precision in procedures; states the meaning of symbols, carefully specifying units of measure and providing accurate labels.
	Forgets to determine the necessary level of precision.	Determines a level of precision; may not be the best choice.	Determines the necessary level of precision; distinguishes between exact and approximate answers and determines which is more useful depending on the context.	Determines the necessary level of precision noting error or uncertainty; distinguishes between exact and approximate answers and determines which is more useful depending on the context.

SMP	Novice	Apprentice	Practitioner	Expert
MP.7 Look for and Make Use of Structure.	Applies an algorithm or method without evaluating their choice; has difficulty using an efficient or appropriate approach.	Uses only one appropriate method or algorithm for a problem even though there may be more efficient approaches.	Analyzes a task and finds more than one method or algorithm to a problem; takes efficiency into consideration.	Uses the most efficient method, algorithm, or solution path for a task based on mathematical structure.
	Has difficulty articulating why they chose an algorithm or solution path.	Analyzes a task before automatically applying an algorithm or solution path; explains why they choose an algorithm or solution path.	Analyzes a task before automatically applying an algorithm or solution path; articulates and justifies their choice of algorithm or solution path.	Analyzes the task before automatically applying an algorithm or solution path; articulates and justifies choice of algorithm or solution path, defends their choice, and questions others' choices.
	Has difficulty recognizing that quantities can be represented in different ways.	Looks for patterns or structures, recognizing that quantities can be represented in different ways; recognizes equivalent forms, but may not be able to choose the best form to use.	Recognizes the significance in concepts and models and use the patterns or structure within a problem to solve related problems; changes structure(s) to an appropriate equivalent form(s) to solve a problem; interprets, models, and connects multiple representations.	Steps back and shifts perspectives based on a problem's structure; identifies similarities between different mathematical forms and uses patterns from known situations to solve nonroutine problems; chooses an equivalent structure based on efficiency or its power to communicate solution.
	Struggles or forgets to analyze reasonableness of intermediate results.	Recognizes when intermediate results do not make sense.	Analyzes reasonableness of intermediate results and if results don't make sense, determines why they don't.	Analyzes reasonableness of intermediate results and changes course to a more effective or efficient process.
	Has difficulty viewing simple quantities both as single objects and as composition of several objects.	Views simple quantities both as single objects or as a composition of several objects and uses operations to make sense of a problem.	View complicated quantities both as single objects or composition(s) of several objects and uses operations to make sense of a problem.	Views and analyses complicated quantities both as single objects or composition(s) of several objects and uses operations flexibly to make sense of a problem.

SMP	Novice	Apprentice	Practitioner	Expert
MP.8 Look for and Express Regularity in Repeated Reasoning.	Examines task in isolation; struggles with understanding the hierarchy within concepts; cannot recognize a known problem within a different context.	Starts to draw connections to prior or future concepts; can complete a similar problem with prompting to get started	Draws connections to prior or future concepts to develop understanding of procedural tasks; can complete a similar problem to get started.	Connects tasks to prior concepts and tasks and evaluates which concepts are most efficient and applies strategies to nonroutine tasks.
	Struggles with demonstrating a logical progression that leads to pattern recognition.	Demonstrates a logical progression that leads to pattern recognition but applies an incorrect progression for the context.	Demonstrates a logical progression that leads to pattern recognition and a solution based on a pattern of repeated reasoning.	Demonstrates a logical progression that leads to pattern recognition and applies solutions based on repeated reasoning to a new situation including nonroutine tasks.
	Struggles with noticing general methods or shortcuts or forgets to look for general methods or shortcuts.	Uses memorized shortcuts without understanding its basis on repeated calculations.	Notices repeated calculations and look for general methods and shortcuts based on repeated calculations.	Generalizes processes and applies them to higher-level problem solving implementing short cuts; generates exploratory questions based on the current task; articulates multiple approaches and/or answers to a problem.
	Struggles to or forgets to monitor intermediate results.	Occasionally monitors intermediate results asking “Why?” when something doesn’t work.	Continually evaluates the reasonableness of intermediate results while attending to the details and make generalizations based on findings.	Continually evaluates the reasonableness of intermediate results while attending to the details and make generalizations based on findings; encourages peers to monitor each other’s intermediate results.

Appendix B: Number Talks Emphasizing Mathematical Structure of Fractions

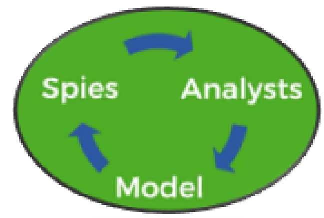
Number talks can be used to help students become more fluent in numeracy and quantitative literacy. There are many resources online to help teachers implement number talk routines in their classrooms. Because of the research that links student understanding of fractions to success in higher level mathematics, Ohio developed a scope and sequence of number talks as part of the Mathematical Modeling and Reasoning course. This scope and sequence can be found at Number Talks Scope and Sequence.

Appendix C: Spies, Analysts, Model Routine

Modeling Process

Modeling is a process that uses mathematics to represent, analyze, make predictions or otherwise provide insight to real-world phenomenon.

General Modeling Principles



- It is usually easier to develop useful models by starting with a simplified version of a situation than with one that is closer to reality. The first model is rarely the final model.
- Pay attention to what you "want." If you need a number, make up a value, but note what you did. That number may become a variable later.
- Be conscious of decisions/assumptions.
- Ask, "What if?" What would happen if (pick a number or assumption) changed?
- Ask, "What question are we trying to answer? How can I 'measure' that?"

Modeling Steps

Step 1: Spies–*What information do I need and how do I acquire it?*
- Do I understand the problem?
- What information do I need?
- How will I acquire it?
- What assumptions am I going to make? What facts or statements am I going to take for granted?

Step 2: Analysts–*Take the information and work with it to figure out how to use it.*
- Decide what information to keep and what if any to discard? Is there other information still needed?
- Create a picture, graph, chart, or other representation (or revise the spies initial model) to help me understand the data?
- Do the math. Show calculations and label units.
- What is your solution? Does it make sense? Is it clear? Does it answer the question? Is the level of precision appropriate?
- Do I need to revise my model and/or solution?

Step 3: Model–*Use the model and verify that it works.*
- How can I communicate my solution to others?
- Can I use my model to make accurate predictions?
- What if our assumptions are wrong? How does that impact our answer?
- What if our scenario changes a little? Do our results change a little or a lot?
- Can my model be generalized to a broader situation? May need to use variables.

Acknowledgements

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Acknowledgements, continued

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